MODIFIED STEFFENSEN TYPE METHOD FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT
Iterative methods for approximation of roots of non-linear equations often breakdown when the derivative of the function value is zero or near zero at the point of iteration. This paper seeks to introduce method that overcomes such breakdown. The method proposed herein is a combination of forward difference formula with Simpson's quadrature in spirit of Steffensen. The convergence analysis of the proposed method is shown to be of order p=2. Numerical experiments are performed and results obtained show the comparative advantage the proposed method has over well-known methods such as variants of Newton's Method.

Keywords: Forward difference formula, Singularity, Zeros, Variants.
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INTRODUCTION

Approximation of roots of non-linear equations

$$f(x) = 0, \tag{1}$$

is still a challenging task in Computational Mathematics, nonlinear equations evolve from most Mathematical models of real life phenomena. Also, solutions to nonlinear equations play important role in convergence analysis of majority of numerical schemes. For example, problems involving computation of eigenvalues of matrices will require roots of characteristics equations [1]. Eigenvalue problems are encountered in the study of stability of Numerical Methods for integrating system of initial value problems for which zeros of the first characteristics polynomials are required, see [1, 2]. The famous scheme for finding roots of Numerical equations is the Newton's method. The Newton's method and its variants fail at the iteration point where:

$$f'(x_n) = 0, \quad n = 0,1,2,\ldots \tag{2}$$

These methods include:
Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

Trapezoidal Method:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n + f(x_n))} \tag{4}$$

Classical Simpson Third Order Method [3]:

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(z_n) + f'(y_n)} \tag{5}$$

where $u_n = \dfrac{f(x_n)}{f'(x_n)}$, (6)

$$y_n = x_n - u_n \tag{7}$$

and

$$z_n = x_n - 2^{-1}u_n \tag{8}$$

New Iterative Method [4]:

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'\left(x_n - \dfrac{f(x_n)}{f'(x_n)}\right) + 4f'\left(x_n - \dfrac{1}{2}\dfrac{f(x_n)}{f'(x_n)}\right) + f'(x_n)} \tag{9}$$

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(z_n) + f'(y_n)} \tag{10}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{11}$$

$$z_n = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)} \tag{12}$$

Methods for overcoming barrier encountered when $f'(x_n) = 0,\ n = 0,1,2,\ldots$ includes the Steffensen's Method

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \tag{13}$$

and its variants. In Steffensen Method, the derivative $f'(x_n)$ is replaced with the forward divided difference operator [5]. The Steffensen Method is of order p=2, same order as the Newton's method from which it is modified. Another modification to overcome singularity problem is contained in [6], wherein, the derivative function is replaced with higher derivative at points where $f'(x_n) = 0,\ n = 0,1,2,\ldots$ is made to the Newton's method, the modification is to replace the derivative $f'(x_n)$ with higher derivative at points where $f'(x_n)=0,\ n=0,1,2,\dots$ and as soon as the singularity points are passed, the algorithm revert to Newton's method to compute the iteration to convergence. The problem with such technique occurs at the point where the discriminant of the square roots is negative and situations where obtaining higher derivative of functions are nonexistent. In this paper, method that overcome breakdown at the points where the derivative of the function is zero or near zero [7, 8] is developed through replacement of all derivative functions in (9). Derivative functions are replaced using forward divided difference operator as employed in Steffensen's method [5]. A derivative free method is also referred as Steffensen's type method [9-11]. This paper is organized as follows: section 2 is on the construction of the method, section 3, the computation order of the proposed method is established, while numerical experiments are carried out in section 4, using standard problems in the literature. Section 5 is the conclusion.

CONSTRUCTION OF METHOD

In this section, the Modified Steffensen's Type Method (MSTM) is presented. The proposed scheme is based on the combination of forward difference formula with Simpson quadrature formulae, [3]. Using the idea in the development of Steffensen's method [5], consider the nonlinear equation (1), with starting value x_0. The derivative $f'(x_n)$ at the point x_0 is given by the forward difference operator as:

$$f'(x_0)\approx\frac{f(x_0+f(x_0))-f(x_0)}{f(x_0)}, \tag{14}$$

where h is the step length.

$x_1=x_0+f(x_0)$

also,

$$h=x_1-x_0=f(x_0). \tag{15}$$

For every x_n, $n=0,1,2,\dots$, the derivative of (1) is given as

$$f'(x_n)\approx\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}=g(x_n) \tag{16}$$

Substituting (16) into (3) gives the Steffensen's method [5]:

$$\frac{f(x_n)}{f'(x_n)}\approx\frac{f(x_n)}{g(x_n)}=\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}, \tag{17}$$

Similarly, substituting (16) into (9), yields

$$x_{n+1}=x_n-\frac{6f(x_n)}{g(x_n)+4g(z_n)+g(y_n)}, \tag{18}$$

where

$$y_n=x_n-\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}, \tag{19}$$

and

$$z_n=x_n-\frac{1}{2}\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}. \tag{20}$$

Also,

$$g(x_n)\approx\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)},$$

$$g(z_n)\approx\frac{f(z_n+f(z_n))-f(z_n)}{f(z_n)}.$$

$$g(y_n)\approx\frac{f(y_n+f(y_n))-f(y_n)}{f(y_n)},$$

The (18) is the proposed method, hereafter referred to as Modified Steffensen's type method (MSTM) for solving (1).

CONVERGENCE ANALYSIS

Proposition 3.1

Let $\alpha\in I$ be a simple zero of sufficiently differentiable function $f:I\to R$ for an open interval I. If x_0 is sufficiently close to α, then the order of convergence p of the methods defined by (18) is of order p=2.

Proof

Let $c_k=\frac{1}{k!}f^k(\alpha)$, and $e_n=x_n-\alpha$.

Since $f(\alpha)$ is sufficiently differentiable To Taylor expand $g(x_n)$, first Taylor expand $f(x_n)$ about α to obtain

$$f(x_n)=e_nc_1+e_n^2c_2+e_n^3c_3+\cdots, \tag{24}$$

Similarly, Taylor's expansion of $f(x_n+f(x_n))$ about α is

$$f(x_n+f(x_n))=e_n(c_1+c_1^2)+e_n^2(c_2+3c_1c_2)+e_n^3(c_3+4c_1c_3+2c_2^2)+\cdots, \tag{25}$$

Subtracting (24) from (25), and dividing the result by (24) gives

$$g(x_n)=e_nc_1^2+e_n^2 2c_1c_2+e_n^3(3c_1c_2+2c_2^2)+\cdots \tag{26}$$

Also, Taylor expand $g(x_n)$ about α, first expand $\dfrac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}$ about x_n to obtain

$$\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}=e_n^2c_1^2+e_n^3c_1c_2+\cdots, \tag{27}$$

substituting (27) into (19) gives

$$y_n-\alpha=e_n-\left(e_n^2c_1^2+e_n^3c_1c_2+\cdots\right). \tag{28}$$

Now, substituting (28) in the Taylor's series of $f(y_n)$ (29)

$$f(y_n)=e_nc_1+e_n^2\left(c_2-c_1^3\right)+e_n^3\left(c_3-3c_1^3c_2\right)+\cdots$$ similarly,

$$f(y_n+f(y_n))=e_nc_1(1+c_1)+e_n^2\left(c_2+c_1^4-c_1^3+c_1^2c_2+2c_1c_2\right)+ e_n^3\left(2c_1^5-6c_1^3+c_1^3c_3+2c_1^2c_2-4c_1^2-c_1c_3+2c_2\right)+\cdots \tag{30}$$

subtracting (29) from (30), gives

$$f(y_n+f(y_n))-f(y_n)=e_nc_1^2+e_n^2\left(c_1^4+c_1^2c_2+2c_1c_2\right)+ e_n^3\left(2c_1^5-4c_1^4-4c_1^3+c_1^3c_3+5c_1^2c_2+c_1^2c_3+2c_2-c_3\right)+\cdots \tag{31}$$

Therefore, the Taylor expansion of (22) become

$$g(y_n)=e_nc_1^2+e_n^2\left(c_1^4+c_1^2c_2+c_1c_2+c_1^3\right)+e_n^3\left(2c_1^6+c_1^4c_2-4c_1^3c_2-c_2^2-c_1c_3\right)+\cdots \tag{32}$$

Similarly, Taylor expansion of (23) follows: Taylor expanding $\dfrac{1}{2}\dfrac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}$ gives

$$\frac{1}{2}\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}=\frac{1}{2}e_n^2c_1^2+\frac{1}{2}e_n^3c_1c_2+\cdots \tag{33}$$

Substituting (33) into (20),

$$z_n-\alpha=e_n-\frac{1}{2}\left(e_n^2c_1^2+e_n^3c_1c_2+\cdots\right), \tag{34}$$

So that,

$$f(z_n)=e_nc_1+e_n^2\left(\tfrac{1}{2}c_2-c_1^3\right)+e_n^3\left(c_3-\tfrac{1}{2}c_1^2c_2-\tfrac{1}{2}c_1^2c_3\right)+\cdots \tag{35}$$

Therefore,

$$f(z_n+f(z_n))=e_nc_1(1+c_1)+e_n^2\left(c_2+\tfrac{1}{2}c_1^4+c_1^3+c_1^2c_2+c_1c_2\right)+ e_n^3\left(\tfrac{1}{2}c_1^2c_2-\tfrac{1}{2}c_1^3c_2+\tfrac{1}{2}c_1^3c_3-3c_1^3-c_1^4+2c_1^2c_2-2c_1^2+2c_1c_3-2c_2+c_3\right)+\cdots \tag{36}$$

Subtracting (35) from (36), results in

$$f(z_n+f(z_n))-f(z_n)=e_nc_1^2+e_n^2\left(c_2(c_1+1)^2+\tfrac{1}{2}c_1^4+c_1^3-c_1c_2-\tfrac{1}{2}c_1^3-c_2\right)+ e_n^3\begin{pmatrix}\tfrac{1}{2}c_1^2c_2-\tfrac{1}{2}c_1^3c_2-\tfrac{1}{2}c_1^3c_3-c_1c_3\\ -3c_1^3-c_1^4+2c_1c_2+2c_1^2+(c_1+1)^3c_3+\tfrac{1}{2}c_1^2c_2+\tfrac{1}{2}c_1^2c_3-c_3\end{pmatrix}+\cdots \tag{37}$$

Then, Taylor's expansion of (23) is given as

$$g(z_n)=e_nc_1^2+e_n^2\left(\tfrac{1}{2}c_1^3+c_1^2c_2\right)+e_n^3(\tfrac{1}{4}c_1^5+\tfrac{1}{2}c_1^4c_2+2c_1^4+6c_1^3-\tfrac{1}{2}c_1^3c_3+\tfrac{3}{2}c_1^3c_2 -4c_1^2c_3+\tfrac{3}{2}c_1^2c_2+4c_1^2+c_1^2c_2+2c_2+4c_1c_2-c_3)+\cdots \tag{38}$$

The Taylor's expansion of the denominator of (18) is therefore

$$g(x_n)+4g(z_n)+g(y_n)=6c_1^2e_n-e_n^2\left(3c_1^3+4c_1^2c_2+4c_1c_2\right)+\cdots \tag{39}$$

Hence, the Taylor expansion of $\frac{6f(x_n)}{g(x_n)+4g(z_n)+g(y_n)}$ about x_n is

$$\frac{6f(x_n)}{g(x_n)+4g(z_n)+g(y_n)} = e_n + e_n^2\left(\frac{2c_2}{3c_1}+\frac{c_1}{2}+\frac{2c_2}{3}\right)+\cdots \tag{40}$$

Substituting (40) into (18),
So as to obtain the error equation

$$e_{n+1} = e_n^2\left(\frac{2c_2}{3c_1}+\frac{c_1}{2}+\frac{2c_2}{3}\right)+O\left(e_n^3\right) \tag{41}$$

Hence, the method is of order p=2.

NUMERICAL EXPERIMENT

In this section, the effectiveness of the new method MSTM is verified, by applying it to nonlinear equations in Table 1.

Table 1: Test Functions and their simple roots

s/n	Nonlinear Equations	Simple Root	Initial point
1	$Cos\ x - x = 0$	$\alpha = 0.739085$	$x_0 = -2$
2	$Sinx - 1 + x = 0$	$\alpha = 0.510973$	$x_0 = 1$
3	$e^x - 3x = 0$	$\alpha = 0.619061$	$x_0 = 0$
4	$x^3 - 6x + 4 = 0$	$\alpha = 0.732051$	$x_0 = 1$
5	$3x^2 - 0.6x - 7 = 0$	$\alpha = -1.4308$	$x_0 = 0.1$
6	$xTan\ x + 1 = 0$	$\alpha = 2.79839$	$x_0 = 2.5$
7	$2Sin\ x - x = 0$	$\alpha = 1.89549$	$x_0 = 2.9$

All computations were carried out using MATHEMATICA 9. For methods considered, the iterations are in successions and converge to an approximate solution of the nonlinear equations. The test equations on Table 1, are used to compare the new method MSTM with other well-known methods: Newton Method (NM), Steffensen's method (SM) and the New Iterative Method (NIM). All the test equations are adapted from [3]. All methods are of order p = 2 except for NIM which is of p = 3. The new method MSTM is similar to the classical methods in operation.

Table 2: Numerical results for the nonlinear equation (1) on Table 1

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)
1	-0.613394	0.734536	7.91330	3.98861
2	0.376528	0.7390892	0.874395	-4.47739
3	0.713909	0.739085	0.736226	0.520357
4	0.738968	-	0.739084	0.728841
5	0.739085	-	0.739085	0.739058
6	-	-	-	0.739085

The proposed Modified Steffensen Type Method (MSTM) is of order p=2 converges faster than the New Iterative Method of order p=3 as shown in Table 2. The oscillatory property of the equation is tracked by both MSTM and NIM.

Table 3: Numerical results for the nonlinear equation (2) on Table 1

Iterates (i)	**Root (x_i) (MSTM)**	**Root (x_i) (NR)**	**Root (x_i) (SM)**	**Root (x_i) (NIM)**
1	0.498677	0.453698	0.265172	0.548969
2	0.510955	0.51058	0.50270	0.511279
3	0.510973	0.510973	0.510948	0.510973
4	-	-	0.510973	-

The rate of convergence of the MSTM as shown in Table 3, is as fast as that of NR and NIM as it has the same number of iterations as the Newton and New Iterative Method but the MSTM converges to the exact root α of the function in each iteration.

Table 4: Numerical results for the nonlinear equation (3) on Table 1

Iterates (i)	**Root (x_i) (MSTM)**	**Root (x_i) (NR)**	**Root (x_i) (SM)**	**Root (x_i) (NIM)**
1	0.816451	0.5	0.553574	Diverges
2	0.638342	0.61006	0.619727	
3	0.619234	0.618997	0.619061	
4	0.619061	0.619061	-	

In Table 4, the Steffensen's converges to α after three iterations while proposed method required additional iteration like the Newton's method. The NIM of order p=3 diverges.

Table 5: Numerical results for the nonlinear equation (4) on Table 1

Iterates (i)	**Root (x_i) (MSTM)**	**Root (x_i) (NR)**	**Root (x_i) (SM)**	**Root (x_i) (NIM)**
1	0.796016	0.666667	0.8	0.790698
2	0.730373	0.730159	0.73857	0.743282
3	0.732071	0.732049	0.732122	0.732721
4	0.732051	0.732051	0.732051	0.732054
5	-	-	-	0.732051

The MSTM approximate the root of the third-degree polynomial (4) at same number of iterations with the Newton's method and Steffensen's method as indicated on Table 5.

In Table 6, the efficiency of the proposed method is demonstrated via its application on problem (5) in Table 1. The Newton method (NR) and NIM diverges while Steffensen's method (SM) and its variants MSTM approximated the root efficiently. Whereas, the Steffensen's Method required ten (10) iterations, the proposed MSTM required nine (9) iterations to converge.

Table 6: Numerical results for the nonlinear equation (5) on Table 1

Iterates (i)	**Root (x_i) (MSTM)**	**Root (x_i) (NR)**	**Root (x_i) (SM)**	**Root (x_i) (NIM)**
1	-0.223119	Diverges	-0.233333	Diverges
2	-0.53067		-0.536487	
3	-0.817694		-0.809952	
4	-1.07353		-1.04799	
5	-1.27705		-1.23887	
6	-1.39654		-1.36606	
7	-1.42901		-1.42127	
8	-1.43079		-1.43056	
9	-1.4308		-1.43079	
10	-		-1.4308	

Table7: Numerical results for the nonlinear equation (6) on Table 1

Iterates (i)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)	Root (x_i) (MSTM)
1	2.77558	2.53062	2.64056	2.33474
2	2.79838	2.6102	2.77668	2.71079
3	2.79839	2.73879	2.79832	2.79672
4	-	2.79680	2.79839	2.79839
5	-	2.79839	-	-

In Table 7, the sinusoidal equation (6) is approximated and the MSTM converges in the same step with the third order NIM.

Table 8: Numerical results for the nonlinear equation (7) on Table 1

Iterates (i)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)	Root (x_i) (MSTM)
1	2.0769	0.852556	-2.68597	1.07413
2	1.91056	3.4634	10.78590	1.58535
3	1.89562	1.26769	1.65651	1.85322
4	1.89549	1.88755	1.87742	1.89466
5	-	1.89547	1.89529	1.89549
6	-	1.89549	1.89549	-

In Table 8, the approximation of the sinusoidal equation (7) reveals that the NIM yielded oscillatory approximation. The proposed MSTM generated stable approximated solution which converged after five iterations.

CONCLUSION

The forward difference equation has been used with the New Iterative Method (NIM) to give a derivative free method for solving nonlinear equations of order p=2. The new method has been compared with methods of order two and three using some numerical examples and the results shows that the proposed MSTM is efficient in approximating nonlinear equations whose first derivative is zero or near zero at the point of iteration.

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