Nigerian Journal of Scientific Research (NJSR)
ISSN-0794-0378

ON THE PARAMETER ESTIMATION OF A NEW INNOVATION DISTRIBUTION FOR GARCH (1,1) MODEL

ODEKINA, O.G.[1]* AND UBAKA, J.N.[2]
[1]Department of Mathematics, Covenant University, Ota, Nigeria, [2]Department of Statistics, University of Lagos, Lagos, Nigeria

ABSTRACT
The autoregressive conditional heteroscedastic (ARCH) and generalized autoregressive conditional heteroscedastic (GARCH) models have been used for modelling the volatility of financial time series data. These models were developed under the assumption of a normal distribution for the innovation distribution. Due to some constraints of the normal distributions like being leptokurtic, some important features of a data cannot be captured. Other distributions for the innovation distribution have been developed over the years. This study proposes a new innovation (error) distribution which was obtained through a transformation function by converting the inverted exponentiated skewed student t distribution into an innovation distribution through a transformation function. The new innovation distribution was used on GARCH (1,1) model in order to obtain the volatility parameters. The volatility parameters were obtained by the method of maximum likelihood estimation. The shape and skewed parameters of this distribution were obtained as well. The proposed error distribution will serve as a competitor to existing error distributions with an improved form of flexibility that captures the attributes of financial assets'volatility behaviour.

Keywords: GARCH, inverted exponentiated skewed student t error distribution, maximum likelihood estimation, volatility
***Correspondence:** Onuche.odekina@covenantuniversity.edu.ng, 08101645043

INTRODUCTION

GARCH modelling under new error distributions signifies an essential development in capturing the behaviour of financial time series. Through the incorporation of distributions whose empirical attributes such as skewness and fat tails are well reflected, researchers can develop a more accurate and robust models. This advancement boosts the ability to comprehend and manage financial risk, thereby leading to improved investment strategies and economic forecasting. The adoption of a new error distribution in GARCH models increases flexibility in modelling financial time series that are complex.

Error distribution is one of the essential techniques in the estimation of parameters of volatility models and Engle [1] proposed the use of the normal distribution as an error distribution in estimating his proposed volatility model. This error distribution has made waves in estimating volatility models. Bollerslev [2] also proposed the adoption of student t distribution which was developed by William [3]. The t-distribution which was adopted by Bollerslev [2] was aimed at addressing the constraint of the normal distribution. Nelson [4] introduced the generalized error distribution which was used for the parameter estimation of an exponential generalized autoregressive conditional heteroscedastic (EGARCH) model. The error distribution which includes the normal, uniform and Laplace distribution was noted to be more robust compared to the student-t distribution [5].

Agboola *et al.* [6] introduced the exponentiated generalized skewed student-t distribution as the underlying distribution for GARCH (1,1) model. The efficiency of the new error distribution was assessed and compared with existing error distributions and it was observed that the exponentiated generalized skewed student-t distribution outperformed other distributions. O'Hagan & Leonard [7] proposed the skewed normal distribution in order to capture the skewness of a normal distribution, it takes into consideration the asymmetric attributes of the normal distribution. Based on the success of O'Hagan & Leonard; Fermandez & Steel [7, 8] adopted the skewed normal distribution as the underlying distribution for innovation distribution on the modelling of volatility.

Theodossiou [9] introduced the skewed generalized error distribution as an error distribution. This was achieved through the modification of a skewed Laplace distribution by the addition of a skewed parameter. The skewed parameter that was added takes into account the skewness of the generalized error distribution and the skewed Laplace distribution. Obalowu & David [10] derived the four parameter odd generalized exponential Laplace distribution, its statistical properties were obtained, the distribution was confirmed to be efficient in modelling financial data. Sequel to the work of

Obalowu & David; Obalowu & David [10, 11] proposed the Odd Generalized Exponential Laplace Distribution as an innovation distribution for GARCH model and its variants. The study by Obalowu & David [10] shows that OGELAD outperform other error distributions considered in their study. Similar to the work of Obalowu & David; OlayemI & Olubiyi [11, 12] proposed the standardized exponentiated Gumbel error distribution as an innovation distribution. Though an explicit solution of the volatility parameters could not be obtained due to the complexity of the resulting equations. However, a numerical solution method was applied to get the parameter values.

The importance of the innovation distribution under volatility modelling cannot be overemphasized as lot of researchers have delved into the applications of existing innovation distributions with little or no improvement on them. This study proposes a new innovation distribution called the Inverted exponentiated skewed student-t error distribution (IESSTED) that will take into account the skewness attributes of financial time series data, the proposed error distribution is more robust and parsimonious with an improved flexibility that caputures the attributes of asset's volatility. The Inverted exponentiated skewed student-t distribution was first introduced by Danjuma *et al.* [13] which consists of two parameters. The parameters of the volatility model for GARCH (1,1) under the inverted exponentiated skewed student-t error distribution were obtained through the method of maximum likelihood estimation.

MATERIALS AND METHODS

Inverted Exponentiated Skewed Student t Distribution

Danjuma *et al.* [13] introduced the inverted exponentiated skewed student t distribution with two parameters whose pdf is given as;

$$f(x; u, \lambda) = \frac{\lambda}{2u(\lambda + x^2)^{3/2}}\left[\frac{1}{2} + \frac{x}{2\sqrt{\lambda + x^2}}\right]^{\frac{1-u}{u}}; \qquad (1)$$

$-\infty > x > \infty, u > 0, \lambda > 0$

The CDF of the above distribution is given as;

$$F(x; u, \lambda) = \left[\frac{1}{2} + \frac{x}{2\sqrt{\lambda + x^2}}\right]^{\frac{1}{u}} \qquad (2)$$

The inverted exponentiated skewed student-t error distribution (IESSTED)

Through the method of transformation function, the proposed error distribution is obtained as follows.

Let $\varepsilon_t = x_t\sigma_t$ be the transformation function [1].

where $\varepsilon_t = x_t\sigma_t$ is the error of the mean equation

Hence, $x_t = \frac{\varepsilon_t}{\sigma_t}$

$$\frac{dx_t}{d\varepsilon_t} = \frac{1}{\sigma_t}$$

$$f(\varepsilon_t; u, \lambda, \sigma_t) = \frac{\lambda}{2u\left[\lambda + \left(\frac{\varepsilon_t}{\sigma_t}\right)^2\right]^{3/2}}\left[\frac{1}{2} + \frac{\frac{\varepsilon_t}{\sigma_t}}{2\sqrt{\lambda + \left(\frac{\varepsilon_t}{\sigma_t}\right)^2}}\right]^{\frac{1-u}{u}} \frac{1}{(\sigma_t{}^2)^{1/2}}; \qquad (3)$$

Equation (3) is the proposed error distribution

Maximum likelihood method of estimation of parameters for GARCH (1, 1)

The likelihood function of the inverted exponentiated skewed student t error distribution (IESSTED) is given as;

$$L(\theta) = \prod_{t=1}^{n}[f(\varepsilon_t; u, \lambda, \sigma_t)]$$

$$L(\theta) = \prod_{t=1}^{n}[f(\varepsilon_t; \theta)]$$

Where $\theta = (u, \lambda, \sigma_t)$

$$L(\theta) = \prod_{t=1}^{n}\left[\frac{\lambda}{2u\left[\lambda + \left(\frac{\varepsilon_t}{\sigma_t}\right)^2\right]^{3/2}}\left[\frac{1}{2} + \frac{\left(\frac{\varepsilon_t{}^2}{\sigma_t{}^2}\right)^{1/2}}{2\sqrt{\lambda + \left(\frac{\varepsilon_t}{\sigma_t}\right)^2}}\right]^{\frac{1-u}{u}} \frac{1}{(\sigma_t{}^2)^{1/2}}\right] \qquad (4)$$

$$L(\theta) = \left[\frac{\lambda}{2u}\right]^n \prod_{t=1}^{n}\left[\frac{1}{\left[\lambda + \left(\frac{\varepsilon_t{}^2}{\sigma_t{}^2}\right)\right]^{3/2}}\left[\frac{1}{2} + \frac{\left(\frac{\varepsilon_t{}^2}{\sigma_t{}^2}\right)^{1/2}}{2\sqrt{\lambda + \left(\frac{\varepsilon_t}{\sigma_t}\right)^2}}\right]^{\frac{1-u}{u}} \frac{1}{(\sigma_t{}^2)^{1/2}}\right] \qquad (5)$$

Taking the log of equation (5), this will give

$$LL(\theta) = n\log\lambda - n\log 2u - \frac{3}{2}\sum_{t=1}^{n}\log\left[\lambda + \left(\frac{\varepsilon_t^2}{\sigma_t^2}\right)\right] + \frac{1-u}{u}\sum_{t=1}^{n}\log\left[\frac{1}{2} + \frac{\left(\frac{\varepsilon_t^2}{\sigma_t^2}\right)^{\frac{1}{2}}}{2\sqrt{\lambda + \left(\frac{\varepsilon_t}{\sigma_t}\right)^2}}\right] - \frac{1}{2}n\log\sigma_t^2 \quad (6)$$

Equation (6) is the general form of the log likelihood of the inverted exponentiated skewed student-t error distribution where u and λ are the shape and skewed parameters respectively, σ_t^2 are the volatility models with vector parameters

$$\vartheta = (\omega, \alpha_1, \alpha_2, \ldots \alpha_i, \beta_1, \beta_2, \ldots \beta_j, \varrho, \gamma)$$

Let σ_t^2 be a GARCH (1,1) model given as $\sigma_t^2 = \omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}$

$$LL(\theta) = n\log\lambda - n\log 2u - \frac{3}{2}\sum_{t=1}^{n}\log\left[\lambda + \left(\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right)\right] + \frac{1-u}{u}\sum_{t=1}^{n}\log\left[\frac{1}{2} + \frac{\left(\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right)^{1/2}}{2\sqrt{\lambda + \frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}}}\right] - \frac{1}{2}n\log(\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}) \quad (7)$$

Taking the partial derivative of equation (7) with respect to $u, \lambda, \emptyset, \omega, \alpha_1$ and β_1 and equating the resulting equations to 0 gives;

$$\frac{dll(\theta)}{du} = \frac{-n}{u} - \frac{1}{u^2}\sum_{t=1}^{n}\log\left[\frac{1}{2} + \frac{\left(\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right)^{1/2}}{2\sqrt{\lambda + \frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}}}\right] = 0 \quad (8)$$

Equation (7) can be expressed as

$$LL(\theta) = n\log\lambda - n\log 2u - \frac{3}{2}n\log\lambda - \frac{3}{2}n\log\left(\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right) + \left(\frac{1-u}{u}\right)n\log\frac{1}{2} + \left(\frac{1-u}{u}\right)n\log\frac{\left(\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right)^{\frac{1}{2}}}{2\sqrt{\lambda + \frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}}} \quad (9)$$

From equation (9),

$$\frac{dll(\theta)}{d\lambda} = \frac{n}{\lambda} - \frac{3n}{\lambda} - \frac{(1-u)}{2u}\frac{n}{\lambda + \left[\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right]} = 0 \quad (10)$$

$$\frac{dll(\theta)}{d\omega} = \frac{3}{2}n\frac{\varepsilon_t^2}{(\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1})^2\left(\lambda + \frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right)} + \left(\frac{1-u}{u}\right)n\frac{dll(\theta)}{d\omega}\left[\frac{1}{2} + \frac{\sqrt{\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}}}{2\sqrt{\lambda + \frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}}}\right] + \frac{1}{2}n\frac{1}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}} = 0 \quad (11)$$

$$\frac{dll(\theta)}{d\alpha_1} = \frac{3}{2}n\frac{\varepsilon_t^4}{(\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1})^2\left(\lambda + \frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}}\right)} + \left(\frac{1-u}{u}\right)n\frac{dll(\theta)}{d\alpha_1}\left[-\frac{\varepsilon_t^4}{2(\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1})^{3/2}}\right] + \frac{1}{2}n\frac{\varepsilon_t^2}{\omega + \alpha_1\varepsilon^2_{t-1} + \beta_1\sigma^2_{t-1}} = 0 \quad (12)$$

$$\frac{dll(\theta)}{d\beta_1} = \frac{3n\varepsilon^2{}_{t-1}\beta_1\sigma^2{}_{t-1}}{(\omega+\alpha_1\varepsilon^2{}_{t-1}+\beta_1\sigma^2{}_{t-1})^2\left(\lambda+\frac{\varepsilon_t{}^2}{\omega+\alpha_1\varepsilon^2{}_{t-1}+\beta_1\sigma^2{}_{t-1}}\right)} + \frac{1-u}{u}n\frac{dll(\theta)}{d\beta_1}\left[-\frac{\varepsilon^2{}_{t-1}\beta_1\sigma^2{}_{t-1}}{(\omega+\alpha_1\varepsilon^2{}_{t-1}+\beta_1\sigma^2{}_{t-1})^{3/2}\sqrt{\lambda+\frac{\varepsilon_t{}^2}{\omega+\alpha_1\varepsilon^2{}_{t-1}+\beta_1\sigma^2{}_{t-1}}}}\right] + \frac{n\beta_1\sigma_{t-1}}{\omega+\alpha_1\varepsilon^2{}_{t-1}+\beta_1\sigma^2{}_{t-1}} \quad (13)$$

DISCUSSION

A new innovation distribution called the inverted exponentiated skewed student-t error distribution (IESSTED) was proposed in this study and applied on the GARCH (1,1) model. The parameters of the model were derived through the method of maximum likelihood by taking a partial derivative of the log likelihood function with respect to each of the parameters to be estimated and equating the resulting equations to 0. Due to the complex nature of the resulting equations, an explicit solution for the paramters could not be obtained. However, the solutions can be obtained through some iterative processes or the use of statistical packages.

CONCLUSION

GARCH modelling under an error distribution implies an indispensable development that captures the characteristics of financial time series.This study delved into a new error distribution called the inverted exponentiated skewed student-t error distribution (IESSTED). A GARCH (1,1) model was considered in this study and the parameters were estimated by the method of maximum likelihood. The new model will be of immense importance in volatility modelling and it will also take into account the asymmetric attributes of a dataset. In addition, the new error distribution will serve as a contribution to the existing literatures on innovation (error) distribution and also, it will be a competitor to other existing error distributions. Moreover, the new model can be used in modelling the volatility of financial time series data such as stock prices and foreign exchange data. Futher research involving a comparative study of the new error distribution with existing ones can be conducted.

REFERENCES

1. ENGLE, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation. *Econometrica*, **50**(4): 987-1008.
2. BOLLERSLEV, T. (1987). A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return. *Review of Economics and Statistics*, **69**(3): 542–547.
3. WILLIAM, S.G. (1908). The probable error of a mean. *Biometrika*, **6**(1): 1-25.
4. NELSON, D. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica*, **59**: 349-370.
5. OLAYEMI, M.S. & OLUBIYI, A.O. (2021). Theoretical Properties of New Error Innovation Distribution on GARCH Model. *American Journal of Theoretical and Applied Statistics*, **10**: 9-13.
6. AGBOOLA, S., DIKKO, H.G. & ASIRIBO, O.E. (2018). On A New Exponentiated Error Innovation Distributions: Evidence of Nigeria Stock Exchange. *Journal of Statistics Applications & Probability*. **2**: 321-331.
7. O'HAGAN, A. & LEONARD, T. (1976). Bayes estimation subject to uncertainty about parameter constraints. *Biometrika*, **63**: 201-2012.
8. FERMANDEZ, C. & STEEL, M.F.J. (1998). On Bayesian modeling of fat tails and skewness. *Journal of the American Statistical Association*, **93**: 359-369.
9. THEODOSSIOU, P. (1998) Financial data and the skewed generalized t distribution. *Management Science*, **44**: 1650- 1661.
10. OBALOWU, J. & DAVID, R. O. (2021). The Odd Generalized exponential Laplace distribution. *39th Annual Conference of the Nigerian Mathematical Society*, 215-230
11. OBALOWU, J. & DAVID, R.O. (2023). GARCH Modelling of Nigeria Stock Exchange Returns with Odd Generalized Exponential Laplace Distribution. *Communication in Physical Sciences*, **9**(4): 572-584.
12. OLAYEMI, M.S. & OLUBIYI, A.O. .(2021). Theoretical Properties of New Error Innovation Distribution on GARCH Model.

American Journal of Theoretical and Applied Statistics, **10**(1): 9-13.
13. DANJUMA, I., DANJUMA, J., OKOLO, A. & EMMANUEL, T. (2024). A two parameter inverted exponentiated skewed student-t distribution: theory and application. *International Journal of Development Mathematics*, **2**: 131-152.